SYMMETRIES OF QUANTUM STRUCTURES

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Peter Šemrl
University of Ljubljana

H Hilbert space

$B(H)$ the algebra of all bounded linear operators on H; M_n when $\dim H < \infty$

$S(H) \subset B(H)$ the real-linear space of all self-adjoint operators; H_n when $\dim H < \infty$

$S(H)^{\geq 0} \subset S(H)$ the set of all positive operators; $H_n^{\geq 0}$ when $\dim H < \infty$

$S(H)^{>0} \subset S(H)$ the set of all positive invertible operators; $H_n^{>0}$ when $\dim H < \infty$

$E(H) \subset S(H)$ the effect algebra, the set of all $A \in S(H)$ satisfying $0 \leq A \leq I$; E_n when $\dim H < \infty$

$P_n(H)$ the set of all rank n projections $\equiv$ the set of all n-dim subspaces, Grassmann space when $\dim H < \infty$

$P_1(H)$ projective space on H

Relations, operations on these sets:

$S(H)$ and the subsets

$\leq$: $\quad A \leq B \iff \langle Ax, x\rangle \leq \langle Bx, x\rangle$

$\circ$: $\quad A \circ B = AB + BA, \quad A \circ B = ABA$

compatibility: $AB = BA$

$E(H)$

$\circ$: $\quad A \circ B = A^{1/2}BA^{1/2}$

$\perp$: $\quad A^{\perp} = I - A$

$\sim$: $\quad A \sim B \iff$

$A = E + F,\ B = E + G,\ \ E + F + G \leq I$

$P_n(H)$

$\|P - Q\|$ operator norm, other norms

Mathematical Physics: $P_1(H)$ the set of pure states of the quantum system

$P, Q \in P_1(H)$: $\mathrm{tr}\,(PQ)$ - transition probability between pure states

$$\|P - Q\| = \sqrt{1 - \mathrm{tr}\,(PQ)}$$

Symmetries on quantum structures:

Bijective maps on $S(H)$ or on $S(H)^{\geq 0}$ or on $E(H)$ or on $P_n(H)$ or...

preserving some of the relations and/or operations mentioned above.

Examples:

- A symmetry on bounded observables with respect to compatibility = (in the language of mathematics) A bijective map
$$\phi : S(H) \to S(H)$$
such that for every $A, B \in S(H)$ we have
$$AB = BA \iff \phi(A)\phi(B) = \phi(B)\phi(A).$$

- A bijective map $\phi : E(H) \to E(H)$ such that

$$A \leq B \iff \phi(A) \leq \phi(B)$$

and/or

$$A \sim B \iff \phi(A) \sim \phi(B)$$

and/or

$$\phi(A \circ B) = \phi(A) \circ \phi(B)$$

and/or

$$\phi(A^{\perp}) = \phi(A)^{\perp}$$

- A bijective map $\phi : P_n(H) \to P_n(H)$ such that

$$\|\phi(P) - \phi(Q)\| = \|P - Q\|$$

$T : H \to H$ conjugate-linear:

$$T(\lambda x) = \overline{\lambda} T x$$

T conjugate-linear and bounded
What is $T^* : H \to H$?

$$\langle Tx, y \rangle = \overline{\langle x, T^* y \rangle}$$

$U : H \to H$ anti-unitary operator:

- U bijective,
- U conjugate-linear,
- $\langle Ux, Uy \rangle = \overline{\langle x, y \rangle}$.

Some results

TH. (Molnár) $\dim H \geq 2$

$$\phi : S(H) \to S(H)$$

bijective, and

$$A \leq B \iff \phi(A) \leq \phi(B).$$

$$\Downarrow$$

$\exists T : H \to H$ bounded bijective linear or conjugate-linear and $S \in S(H)$:

$$\phi(A) = TAT^* + S$$

WLOG: $\phi(0) = 0$, then $S = 0$, then ϕ automatically real-linear!

TH. (Ludwig) $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

bijective, and

$$A \leq B \iff \phi(A) \leq \phi(B)$$

and

$$\phi(A^{\perp}) = \phi(A)^{\perp}$$

$$\Downarrow$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

TH. (Gudder, Greechie) $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

bijective, and

$$\phi(A^{1/2}BA^{1/2}) = \phi(A)^{1/2}\phi(B)\phi(A)^{1/2}$$

$$\Downarrow$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

TH. (Molnár) $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

bijective, and

$$A \leq B \iff \phi(A) \leq \phi(B)$$

and

$$A \sim B \iff \phi(A) \sim \phi(B)$$

$$\Downarrow$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

TH (Jamison, Botelho, Molnár, Geher, S)
H real or complex Hilbert space, $\dim H > n$.
$\phi : P_n(H) \to P_n(H)$ surjective.

$$\|P-Q\| = \|\phi(P)-\phi(Q)\|, \quad P, Q \in P_n(H).$$

If $\dim H \neq 2n$, then $\exists$ unitary or antiunitary $U : H \to H$:

$$\phi(P) = UPU^*.$$

If $\dim H = 2n$ we have either the above or

$$\phi(P) = U(I-P)U^*.$$

Moreover, if $\dim H < \infty$, then no surjectivity assumption is needed.

Improvements? Optimal Versions?

Ludwig: bijective → surjective (surjectivity essential)

TH. ??? $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

bijective (surjective), and

$$A \leq B \Rightarrow \phi(A) \leq \phi(B)$$

and

$$\phi(A^\perp) = \phi(A)^\perp$$

$$\Downarrow \; ???$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

No. Optimality!

Open problems:

- What is the general form of bijective maps $\phi : E(H) \to E(H)$ satisfying
$$A \le B \Rightarrow \phi(A) \le \phi(B)$$
and
$$\phi(A^\perp) = \phi(A)^\perp?$$

- The finite-dimensional case?
- Under the continuity assumption?
- Other variations (see the next page)?

$2 \leq \dim H < \infty$

$\phi : E(H) \to E(H)$ continuous

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\phi(A^{\perp}) = \phi(A)^{\perp}.$$

$$\Downarrow$$

$\exists$ a bijective linear or conjugate-linear operator $T : H \to H$ satisfying $\|T\| \leq 1$:

$$\phi(A) = \frac{1}{2}(I - TT^*) + TAT^*.$$

TH. $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

surjective, and

$$A \leq B \Rightarrow \phi(A) \leq \phi(B)$$

and

$$A \sim B \iff \phi(A) \sim \phi(B)$$

$$\Downarrow$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

TH. $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

surjective, and

$$A \leq B \iff \phi(A) \leq \phi(B)$$

and

$$A \sim B \Rightarrow \phi(A) \sim \phi(B)$$

$$\Downarrow$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

TH. ??? $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

surjective, and

$$A \leq B \Rightarrow (A) \leq \phi(B)$$

and

$$A \sim B \Rightarrow \phi(A) \sim \phi(B)$$

$$\Downarrow \text{ ???}$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

No. Optimality

Open questions as above.

H separable infinite-dimensional

$P_\infty(H)$ the set of all projections whose images and kernels are both infinite-dimensional

TH. H an infinite-dimensional complex (real) separable Hilbert space,
$\phi\colon P_\infty(H) \to P_\infty(H)$ a surjective map

$$\|\phi(P)-\phi(Q)\| = \|P-Q\|, \quad P, Q \in P_\infty(H).$$

Then $\exists$ a unitary or an antiunitary operator (orthogonal operator) U on H such that either

$$\phi(P) = UPU^*, \quad P \in P_\infty(H);$$

or

$$\phi(P) = U(I-P)U^*, \quad P \in P_\infty(H).$$

TH. $n \geq 3$, $\phi : E_n \to E_n$ continuous

$$\phi(A^{1/2}BA^{1/2}) = \phi(A)^{1/2}\phi(B)\phi(A)^{1/2}$$

Then we have one of the following seven possibilities:

- there exist a unitary $n \times n$ matrix U and a non-negative real number c such that

$$\phi(A) = (\det A)^c UAU^*;$$

- there exist a unitary $n \times n$ matrix U and a non-negative real number c such that

$$\phi(A) = (\det A)^c UA^tU^*;$$

- there exists a unitary $n \times n$ matrix U such that

$$\phi(A) = U(\operatorname{adj} A)U^*;$$

- there exists a unitary $n \times n$ matrix U such that

$$\phi(A) = U(\operatorname{adj} A)^tU^*;$$

- there exist a unitary $n \times n$ matrix U and a real number $c > 1$ such that

$$\phi(A) = (\det A)^c U A^{-1} U^*$$

if A is invertible, and $\phi(A) = 0$ otherwise;

- there exist a unitary $n \times n$ matrix U and a real number $c > 1$ such that

$$\phi(A) = (\det A)^c U (A^{-1})^t U^*$$

if A is invertible, and $\phi(A) = 0$ otherwise;

- there exist a nonnegative integer $m \leq n$, nonnegative real numbers $c_1, \ldots, c_m$, and a unitary matrix U such that

$$\phi(A) =$$

$$U \operatorname{diag}((\det A)^{c_1}, \ldots, (\det A)^{c_m}, 0, \ldots, 0) U^*.$$

Two main problems

Compare pages 8 and 9

Obvious conjecture:

CONJ. $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

bijective and

$$A \leq B \iff \phi(A) \leq \phi(B).$$

$$\Downarrow$$

$\exists U : H \to H$ unitary or anti-unitary:

$$\phi(A) = UAU^*$$

Motivated by the result on page 11:

PR. $\dim H \geq 2$

$$\phi : E(H) \to E(H)$$

bijective and

$$A \sim B \iff \phi(A) \sim \phi(B).$$

$$\Downarrow$$

???

Example:

$$A \mapsto$$

$$S^{-1/2}\left((I - T^2 + T(I+A)^{-1}T)^{-1} - I\right)S^{-1/2}$$

$$S = \frac{T^2}{2I - T^2}$$

Operator intervals: $A, B \in S(H)$, $A < B$

$$[A, B] = \{C \in E(H) \, : \, A \le C \le B\}$$

$$E(H) = [0, I]$$

Bijective maps preserving order in both directions:

$$[A, B] \to [A + C, B + C]$$

$$X \mapsto X + C$$

$$[A, B] \to [TAT^*, TBT^*]$$

$$X \mapsto TXT^*$$

Bijective map satisfying $X \leq Y \iff \phi(Y) \leq \phi(X)$:

$0 < A < B$

$$[A, B] \to [B^{-1}, A^{-1}]$$

$$\phi(X) = X^{-1}$$

$$[0, I] \to [0, I]$$

$$\phi(X) = I - X$$

$$A \mapsto I + A \mapsto (I + A)^{-1} \mapsto$$

$$T(I+A)^{-1}T \mapsto I - T^2 + T(I+A)^{-1}T \mapsto$$

$$(I - T^2 + T(I + A)^{-1}T)^{-1} \mapsto$$

$$(I - T^2 + T(I + A)^{-1}T)^{-1} - I$$

p a real number, $p < 1$.

$$f_p : [0,1] \to [0,1]$$

$$f_p(x) = \frac{x}{px + (1-p)}, \quad x \in [0,1].$$

TH. $\dim H \geq 3$. $\phi : E(H) \to E(H)$ bijective.

$$A \leq B \iff \phi(A) \leq \phi(B)$$

$$\Downarrow$$

$\exists p, q \in (-\infty, 1)$, $\exists$ a bijective linear or conjugate-linear bounded operator $T : H \to H$ with $\|T\| \leq 1$:

$$\phi(A) =$$

$$= f_q\left((f_p(TT^*))^{-1/2} \, f_p(TAT^*) \, (f_p(TT^*))^{-1/2} \right).$$

TH. $\dim H \geq 2$
$\phi\colon E(H) \to E(H)$ bijective

$$A \sim B \iff \phi(A) \sim \phi(B)$$

$$\Downarrow$$

$\exists$ unitary or antiunitary $U\colon H \to H$, and
a bijective $g\colon [0,1] \to [0,1]$:

$$\{\phi(A), \phi(A^\perp)\} = \{UAU^*, UA^\perp U^*\}$$

for A not scalar, and

$$\phi(tI) = g(t)I, \quad t \in [0,1].$$

Some explanation:

$$A^{\sim} := \{C \in E(H) \colon C \sim A\}$$

TH. $A \in E(H)$, $P \in P(H)$. Then:

$A^{\sim} = E(H)$ if and only if $A \in Sca(H)$.

TH. $A, B \in E(H)$. Then:

(i) $A \sim B$,

(ii) there exist effects $M, N \in E(H)$ such $M \leq A$, $N \leq I - A$, and $M + N = B$.

$A, B \in E(H)$. TFAE:

(i) $B \in \{A, A^{\perp}\}$ or $A, B \in Sca(H)$,

(ii) $A^{\sim} = B^{\sim}$.

Ideas - the finite dimensional case

$$H_n, \qquad E_n$$

$$A, B \in H_n \ \text{ adjacent}$$
$$\Updownarrow$$
$$\operatorname{rank}(A - B) = 1$$

$\phi : H_n \to H_n$ preserves adjacency in both directions, if

$$A, B \ \text{adj} \iff \phi(A), \phi(B) \ \text{adj}$$

$$M = \{(x, y, z, t) \,:\, x, y, z, t \in \mathbf{R}\}$$

$$(x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2) \in M \quad \text{coherent}$$
$$\Updownarrow$$
$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = c^2(t_1-t_2)^2$$

In mathematical foundations of relativity we usually use the harmless normalization $c = 1$.

Two space-time events are coherent (light-like) $\iff$ a light signal can be sent from one to the other

Alexandrov: description of bijective maps on M preserving coherency in both directions

$$r = (x, y, z, t) \leftrightarrow \begin{bmatrix} t+z & x+iy \\ x-iy & t-z \end{bmatrix} = A$$

$$A \in H_2$$

$$\det A = t^2 - z^2 - x^2 - y^2$$

$$r_1, r_2 \in M, \quad r_j \leftrightarrow A_j$$

$$r_1, r_2 \text{ coherent } \iff \det(A_2 - A_1) = 0$$
$$\Updownarrow$$
$$A_2 - A_1 \text{ singular}$$
$$\Updownarrow$$
$$A_1 = A_2 \text{ or } A_1 \text{ and } A_2 \text{ adjacent}$$

Thus, Alexandrov problem = study of adjacency preservers on H_2

$A, B \in H_n$, $A \neq B$. TFAE:

- A, B adj.
- A, B comparable and if C, D belong to operator interval between A and B, then C and D comparable.

Proof. ($\Downarrow$)
$B = A + tP$, say $t > 0 \Rightarrow A \leq B$

$$[A, B] = \{A + sP \ : \ 0 \leq s \leq t\}$$

$C, D \in [A, B] \Rightarrow C = A + s_1 P, D = A + s_2 P.$

($\Uparrow$) A, B not adjacent
If A, B not comparable, done.
If comparable, WLOG $A \leq B$. rank $(B - A) \geq 2 \Rightarrow$ "enough room" to find two noncomparable in $[A, B]$.

L.-K. Hua (1945-1951) FTGM's (motivated by some problems in geometry)

THEOREM. $\phi : H_n \to H_n$, bij., adj $\leftrightarrow$

$$\Downarrow$$

$$\phi(A) = \pm TAT^* + S$$

or

$$\phi(A) = \pm TA^tT^* + S$$

WLOG: $\phi(0) = 0 \Rightarrow S = 0$

Reduction to 2×2 case (=Minkowski!!)

$$\phi : H_n \to H_n$$

ϕ bijective, ϕ preserves adjacency in both directions

$$\text{WLOG}: \quad 0 \mapsto 0$$

$$\text{rank one} \mapsto \text{rank one}$$

$$\text{rank two} \mapsto \text{rank two}$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \begin{bmatrix} * & * \\ * & * \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} * & * \\ * & * \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_2 \to H_2$$

local to global

P, Q orthogonal rank one projections on a three-dimensional Hilbert space:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If R is a rank one projection such that $\|R - P\| = \|R - Q\| \leq \frac{1}{\sqrt{2}}$, then

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{it} & 0 \\ \frac{1}{2}e^{-it} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$M(P,Q) =$$

$$\left\{ R \in P_n(H) \,:\, \|R-P\| \leq \frac{1}{\sqrt{2}} \text{ and } \|R-Q\| \leq \frac{1}{\sqrt{2}} \right\}$$

$P \perp Q \Rightarrow$

$$P = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$M(P,Q) =$$

$$= \left\{ \begin{bmatrix} \frac{1}{2}I_n & \frac{1}{2}U & 0 \\ \frac{1}{2}U^* & \frac{1}{2}I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} : U \in \mathcal{U}_n \right\}.$$

In particular, $M(P,Q)$ is a compact manifold.

$P, Q \in P_n(H)$, $\|P - Q\| = 1$, $\dim H \geq 2n$. Then

$P \perp Q \iff M(P,Q)$ compact manifold

ϕ preserves orthogonality.

Chow's fundamental theorem of geometry of Grassmann spaces

Characterizing adjacency by orthogonality: $P \neq Q$, P, Q adjacent $\Rightarrow$ the orthogonal complement of P, Q large $\Rightarrow$ the orthogonal complement of the orthogonal complement small.

ϕ isometry $\Rightarrow$ ϕ preserves orthogonality $\Rightarrow$ ϕ preserves adjacency! Apply Chow

Problem: Surjective isometries of $P_\infty(H)$ do not preserve orthogonality!

$$H = K \oplus K \oplus K$$

$\phi\colon P_\infty(H) \to P_\infty(H)$ a bijective isometry defined by $\phi(P) = I - P$, $P \in P_\infty(H)$.

$$P = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are orthogonal, while

$$\phi(P) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and

$$\phi(Q) = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

are not orthogonal.

P, Q orthogonal rank one projections on a three-dimensional Hilbert space:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{it} & 0 \\ \frac{1}{2}e^{-it} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$\gamma(\theta) =$$

$$= \begin{bmatrix} (\cos^2\theta) & (\cos\theta\sin\theta)e^{it} & 0 \\ (\cos\theta\sin\theta)e^{-it} & (\sin^2\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the unique path from P to Q passing through R such that

$$\|\gamma(\theta_1) - \gamma(\theta_2)\| = \sin|\theta_1 - \theta_2|$$

for all $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$.

$$P, Q \in P_\infty(H):$$

$P \sim Q \iff P \perp Q$ or $I - P \perp I - Q$

ϕ isometry $\Rightarrow \phi$ preserves $\sim$.

$A, B \in E(H)$, $A \prec B \iff$

$\forall C \in A^{\sim} \setminus \mathrm{Sc}\,(H) \quad \exists D \in B^{\sim} \setminus \mathrm{Sc}\,(H):$

$C^{\sim} \subseteq D^{\sim}$.

Clearly: $B \prec B$ and $B^{\perp} \prec B$.

$A \in E(H) \setminus \mathrm{Sc}\,(H)$. TFAE:

- $A \in P(H)$,
- $\mathrm{card}\,\{B \in E(H) \setminus \mathrm{Sc}\,(H) \colon B \prec A\} = 2$.

$A \notin P(H)$:

We shall construct a non-trivial projection P (which is obviously different from both A and $A^{\perp}$) such that $P \prec A$.

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

$$\sigma(A_1) \subset [0, \varepsilon]$$
$$\sigma(A_2) \subset [\varepsilon, 1 - \varepsilon]$$
$$\sigma(A_3) \subset [1 - \varepsilon, 1]$$

$$P = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Our goal: $P \prec A$.

C a non-scalar effect, $C \sim P$.

Then C and P commute.

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12}^* & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}$$

$$D := \varepsilon \cdot C$$

$$\varepsilon \cdot \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12}^* & C_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq I - A$$

and

$$\varepsilon \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \leq A.$$

Commutativity preservers on $P(H)$.

TH. $\dim H \geq 3$.
$\phi : P(H) \to P(H)$ bijective.

$$PQ = QP \quad \Longleftrightarrow \quad \phi(P)\phi(Q) = \phi(Q)\phi(P)$$

$$\Downarrow$$

$$\phi(P) \in \{UPU^*, UP^\perp U^*\}$$

Let P and Q be two arbitrary commuting projections. Then

$$P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\{P,Q\}'' = (\{P,Q\}')'$$

$$= \left\{ \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 \\ 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & R_4 \end{bmatrix} : R_j \in P(H_j) \right\}'$$

$$= \left\{ \begin{bmatrix} \lambda_1 I & 0 & 0 & 0 \\ 0 & \lambda_2 I & 0 & 0 \\ 0 & 0 & \lambda_3 I & 0 \\ 0 & 0 & 0 & \lambda_4 I \end{bmatrix} : \lambda_j \in \{0,1\},\ j = 1,2,3,4 \right\}$$

$P \notin \{0, I\}$ is of rank one or corank one

$$\iff$$

$$\forall Q,\ QP = PQ\ :\ \text{card}\,\{P, Q\}'' \leq 8$$

Wigner