Inverse Problems in Algebra and Topology II
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Inverse Problems in Algebra

Inverse Galois Problem

Does every finite group occur as the Galois group of an irreducible polynomial over $\mathbb{Q}$?

Inverse Problem for Automorphism Groups

Does every finite group occur as $\mathrm{Aut}(G)$ for some finite group G?

- The Inverse Galois Problem is an open question.
- For the Inverse Problem for Automorphism Groups, we have the following negative result:

Theorem

The cyclic group $\mathbb{Z}_{2n+1}$ is not realized as an automorphism group $\mathrm{Aut}(G)$ for G finite and $n \geq 1$.

Inverse Problem for Graph Automorphisms

Definition

A simple graph Γ is a set vertices V together with a collection of unordered edges $\{v_1, v_2\}$. A *graph automorphism* is a bijection $f : V \to V$ such that $\{f(v_1), f(v_2)\}$ is in E if and only if $\{v_1, v_2\}$ is in E.

Question

Is every finite group G isomorphic to $\mathrm{Aut}(\Gamma)$ for some simple graph Γ?

Answer

Yes!

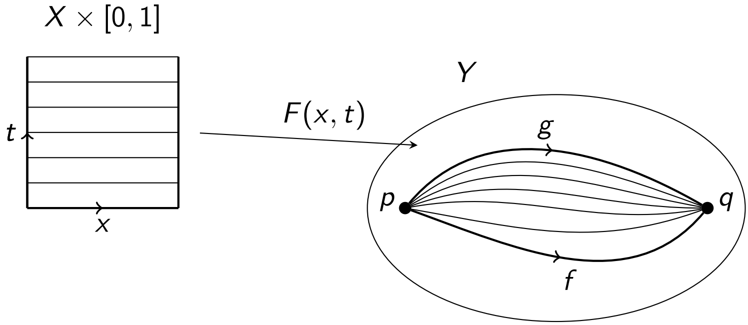
Theorem (Frucht, 1939)

Given a finite group G there exists a simple graph Γ with $\mathrm{Aut}(\Gamma) \cong G$.

Homotopy Theory

Definition (Homotopy of Maps)

Given $f, g \colon X \to Y$ = continuous functions with $f(x_0) = g(x_0) = y_0$. We say f and g are *homotopic* if there is a map $F \colon X \times [0,1] \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$ and $F(x,t) = y_0$ for all $t \in [0,1]$.



Homotopy Groups

$[X, Y]$ = homotopy equivalence classes of $f: X \to Y$ with $f(x_0) = y_0$
$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ = the n-sphere.

Fundamental Group

$\pi_1(X, x_0) = [S^1, X]$ = group of loops in X at x_0.

Example: $\pi_1(S^1, 1) \cong \mathbb{Z}$.

We say a space X is *simply connected* if $\pi_1(X, x_0) = 0$.

Examples: The n-spheres S^n are simply connected for $n \geq 2$.

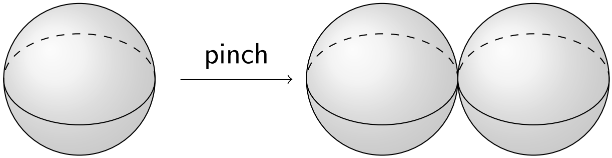
Higher Homotopy Groups

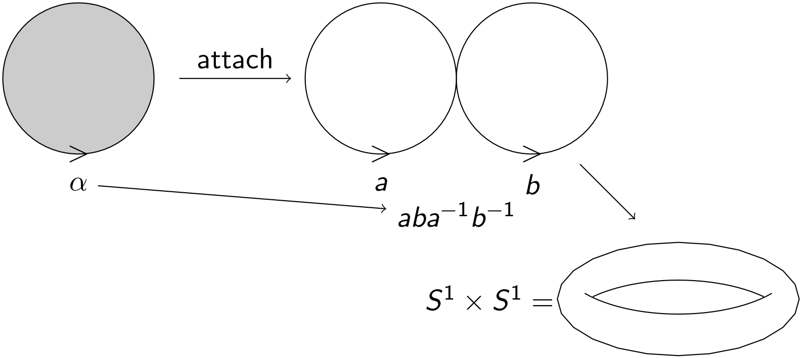
$\pi_n(X, x_0) = [S^n, X]$ = the nth homotopy group of X

Theorem

The groups $\pi_n(X, x_0)$ *are abelian for* $n \geq 2$.

Two Maps with Spheres





Structure of Homotopy Groups

sum: $\alpha, \beta \in \pi_n(X) \implies \alpha + \beta \in \pi_n(X)$

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\alpha \vee \beta} X \vee X \xrightarrow{\text{fold}} X.$$

$$S^n \xrightarrow{\alpha+\beta} X$$

bracket: $\alpha \in \pi_m(X), \beta \in \pi_n(X) \implies [\alpha, \beta] \in \pi_{m+n-1}(X)$

$$S^{m+n-1} \xrightarrow{\text{attach}} S^m \vee S^n \xrightarrow{\alpha \vee \beta} X \vee X \xrightarrow{\text{fold}} X.$$

$$S^{m+n-1} \xrightarrow{[\alpha,\beta]} X$$

Cohomology

The *cohomology* $H^*(X;\mathbb{Z})$ of a space X is a *graded ring*:

$$\alpha, \beta \in H^n(X;\mathbb{Z}) \implies \alpha + \beta \in H^n(X;\mathbb{Z}).$$

cup product

$$\alpha \in H^m(X;\mathbb{Z}), \beta \in H^n(X;\mathbb{Z}) \implies \alpha \cdot \beta \in H^{m+n}(X;\mathbb{Z}).$$

anticommutativity

$$\alpha \cdot \beta = (-1)^{mn} \beta \cdot \alpha$$

Taking coefficients in $\mathbb{Q}$, we obtain a *graded algebra* $H^*(X;\mathbb{Q})$.

Free Graded Algebra

Given $V = \bigoplus_{n \geq 0} V_n$ a graded vector space,

Write $V = \langle x_1, \dots, x_m, y_1, \dots, y_n \rangle$ for $|x_k|$ even and $|y_j|$ odd.

Free Graded Algebra

$$\wedge V = P(x_1, \dots, x_m) \otimes E(y_1, \dots, y_n)$$

Eilenberg-Mac Lane Spaces

Question

Given an abelian group G and $n \geq 2$ is there a space X with $\pi_n(X) \cong G$ and $\pi_k(X) = \{e\}$ for $k \neq n$?

Theorem (Eilenberg-Mac Lane)

Yes! There is such a space $X = K(G, n)$.

Loop Space

Given a space X with a basepoint $x_0 \in X$ define the *loop space* of X by

$$\Omega X = \mathrm{map}((S^1, 1), (X, x_0))$$

Concatenation of loops based at x_0 makes ΩX a *group-like space* with associativity, inverses, and identity all holding up to homotopy.

Theorem

For G abelian and $n \geq 2$ we have $K(G, n) = \Omega K(G, n-1)$.

Two Inverse Problems in Homotopy Theory

Definition (Self-Homotopy Equivalences)

Given a space X we define

$$\mathrm{Aut}(X) = \{f\colon X \to X \mid f \text{ homotopy equivalence } \} \subseteq \mathrm{map}(X, X)$$
$$\mathcal{E}(X) = \text{path components of } \mathrm{Aut}(X)$$

Theorem

The space $\mathrm{Aut}(X)$ is a group-like space and the set $\mathcal{E}(X)$ is a group under composition of functions.

Inverse Problem for $\mathcal{E}(X)$

Does every finite group G occur as $G \cong \mathcal{E}(X)$ for some space X?

Inverse Problem for $\mathrm{Aut}(X)$

Does every loop space ΩY occur as $\Omega Y \simeq \mathrm{Aut}(X)$ for some space X?

Work on the group $\mathcal{E}(X)$

Classical Results:

- $\mathcal{E}(S^n) \cong \mathbb{Z}_2$
- $\mathcal{E}(K(\pi, n)) = \mathrm{Aut}(\pi)$

Early Calculations

- [Olum, 1965] For $q > 2$, $\mathcal{E}(L(p, q)) = \{x \in \mathbb{Z}_q^* | x^2 = \pm 1\}$
- [Oka-Sawashita-Sugawara, 1974] $\mathcal{E}(Sp(2)) = \mathrm{D}_{120}$

Realization Results:

- [Oka, 1980] For n odd there is a space X_n with $\mathcal{E}(X_n) \cong \mathbb{Z}_n$
- [Maruyama, 1994] There is a space X with $\mathcal{E}(X) \cong \mathbb{Z}$.

Rational Homotopy Theory

Rationalization

Let X be simply connected of finite type. There exists a space $X_{\mathbb{Q}}$ and a map $e\colon X \to X_{\mathbb{Q}}$ with e inducing isomorphisms:

$$H^*(X;\mathbb{Q}) \cong H^*(X_{\mathbb{Q}};\mathbb{Z}) \quad \text{and} \quad \pi_*(X) \otimes \mathbb{Q} \cong \pi_*(X_{\mathbb{Q}}).$$

Theorem (Sullivan, 1977)

Let X be simply connected of finite type. There exists a differential graded algebra $\mathcal{M}(X) = \wedge V, d$ over $\mathbb{Q}$ satisfying:

- $d\colon \wedge^q V \to \wedge^{q+1} V$ *with* $d^2 = 0$ *and* $d(V) \subseteq \wedge^+ V \cdot \wedge^+ V$.
- $H^*(\wedge V, d) \cong H^*(X;\mathbb{Q})$.
- $V \cong \pi_*(X) \otimes \mathbb{Q}$.
- *The Sullivan minimal model $\mathcal{M}(X)$ is a complete and faithful invariant of $X_{\mathbb{Q}}$.*

Sullivan Model Examples

space	Sullivan model $\wedge V, d$
S^{2n+1}	$\wedge(x), d$ with $\lvert x\rvert = 2n+1$ and $d = 0$
S^{2n}	$\wedge(x,y), d$ with $\lvert x\rvert = 2n, \lvert y\rvert = 4n-1$ and $dx = 0, dy = x^2$.
$\mathbb{C}P^n$	$\wedge(x,y), d$ with $\lvert x\rvert = 2, \lvert y\rvert = 2n+1$ and $dx = 0, dy = x^{n+1}$.
G	$\wedge(V), d$ with $V \cong \pi_*(G) \otimes \mathbb{Q}$ and $d = 0$.
G/H $\mathrm{rk} G = \mathrm{rk} H$	$\wedge(x_1, \ldots, x_n) \otimes \wedge(y_1, \ldots, y_n), d$ $\lvert x_i\rvert$ even $\lvert y_j\rvert$ odd, $d(x_i) = 0$ and $d(y_j) = P_j(x_1, \ldots, x_n)$.

The Group $\mathcal{E}(X_{\mathbb{Q}})$

Definition (DGA Homotopy)

We say two DGA maps $f, g\colon \wedge V, d \to \wedge W, d$ are *DGA homotopic* written $f \simeq_{\mathrm{DGA}} g$ if there is a DGA map

$$\wedge V, d \xrightarrow{\quad F \quad} (\wedge W, d) \otimes \wedge(t, dt) \ \text{ with } \pi_0 F = f, \pi_1 F = g$$

where $\pi_0(t) = 0$ and $\pi_1(t) = 1$.

Theorem (Sullivan, 1977)

Let X be simply connected and of finite type. Then

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathrm{Aut}_{\mathrm{DGA}}(\wedge V, d)/\simeq_{\mathrm{DGA}} .$$

Theorem (Sullivan-Wilkerson, 1977)

If X is a finite CW complex then $\mathcal{E}(X_{\mathbb{Q}})$ is finitely presented.

Some Realization Results for $\mathcal{E}(X_{\mathbb{Q}})$

- [Arkowitz-Lupton, 2000] There are (non-trivial) rational spaces $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ with

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \{e\} \text{ and } \mathcal{E}(Y_{\mathbb{Q}}) \cong \mathbb{Z}_2$$

- [Federinov-Félix, 2007] Let $\mathcal{E}_\sharp(X) \subseteq \mathcal{E}(X)$ denote the subgroup of self-equivalences inducing the identity on homotopy groups. All 2-solvable nilpotent groups G are realized as $\mathcal{E}_\sharp(X_{\mathbb{Q}})$ for some rational space $X_{\mathbb{Q}}$.

- [Benkhalifa, 2009] Let $n < 10$. Then there is a rational space $X_{\mathbb{Q}}$ with

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathbb{Z}_2^n.$$

The Costoya-Viruel Construction

Theorem (Costoya-Viruel, 2015)

Let G be a finite group. Then there is a space X with $\mathcal{E}(X_{\mathbb{Q}}) \cong G$.

Sketch of Proof

Step 1. Arkowitz-Lupton Rigid Model

$$(\wedge(Z), d) \qquad Z = \langle x_1, x_2, y_1, y_2, y_3, z\rangle$$

$$\begin{aligned}
|x_1| &= 8 & d(x_1) &= 0\\
|x_2| &= 10, & d(x_2) &= 0\\
|y_1| &= 33, & d(y_1) &= x_1^3x_2\\
|y_2| &= 35, & d(y_2) &= x_1^2x_2^2\\
|y_3| &= 37, & d(y_3) &= x_1x_2^3\\
|z| &= 119, & d(z) &= y_1y_2x_1^4y_1^4 + y_2y_3x_1^6 + x_1^{15} + x_2^{12}
\end{aligned}$$

Theorem: $\mathcal{E}(\wedge(Z), d)) \cong \{e\}$.

Step 2. Costoya-Viruel Construction

Let G be a given finite group and $\Gamma = (V, E)$ a graph with $\mathrm{Aut}(\Gamma) \cong G$.

$$(\wedge(W), d) \qquad W = \langle x_1, x_2, y_1, y_2, y_3, z\rangle \oplus \langle x_v, z_v \mid v \in V\rangle$$

$$\begin{aligned}
|x_1| &= 8 & d(x_1) &= 0\\
|x_2| &= 10, & d(x_2) &= 0\\
|y_1| &= 33, & d(y_1) &= x_1^3 x_2\\
|y_2| &= 35, & d(y_2) &= x_1^2 x_2^2\\
|y_3| &= 37, & d(y_3) &= x_1 x_2^3\\
|x_v| &= 40, & d(x_v) &= 0\\
|z| &= 119, & d(z) &= y_1 y_2 x_1^4 y_1^4 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12}\\
|z_v| &= 119, & d(z_v) &= x_v^3 + \sum_{(v,w)\in E} x_v x_w x_2^2
\end{aligned}$$

Theorem: $\mathcal{E}(\wedge(W), d)) \cong \mathrm{Aut}(\Gamma) \cong G$.

Cayley's Theorem and $\mathcal{E}(X)$

Theorem (Cayley)

Every finite group G is isomorphic to a subgroup of S_n.

Benkhalifa, 2017

Given a finite group $G = \{e, g_2, \ldots, g_n\}$ let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ denote the corresponding subgroup of S_n. There is a DG algebra model

$$(\wedge(W), d) \qquad W = \langle x_1, x_2, y_1, y_2, y_3 \rangle \oplus \langle w_j, z_j \mid j = 1, \ldots, n \rangle$$

with differential d extending the Costoya-Viruel differential by

$$dw_j = 0 \qquad d(z_j) = w_j^3 + \sum_{t=1}^{n} w_j w_{\sigma_t}(t) x_2^4 + x_1^{15} + u$$

Benkhalifa Construction continued

Theorem

There is a short exact sequence

$$0 \to H^{119}(\wedge V, d) \to \mathcal{E}(X_{\mathbb{Q}}) \to D^{119}_{40} \to 0.$$

Theorem

$D^{119}_{40} \cong G$ *and* $H^{119}(\wedge V, d) = 0$.

Corollary

$G \cong \mathcal{E}(\wedge V, d) \cong D^{119}_{40}$

Corollary

There is a Sullivan model $\wedge V, d$ *(not of finite type) with* $\mathcal{E}(\wedge V, d) \cong \mathbb{Z}$.

$\mathrm{Aut}(X)$ and its classifying space $\mathrm{BAut}(X)$

Theorem (Dold-Lashof, 1956)

The space $\mathrm{Aut}(X)$ *has a classifying space* $\mathrm{BAut}(X)$ *with*

$$\Omega\mathrm{BAut}(X) \simeq \mathrm{Aut}(X)$$

Definition

A fibration $p\colon E \to B$ with fibre $p^{-1}(b_0) \simeq X$ is called an *X-fibration.*

Theorem (Stasheff, Dold, May)

Let X be a CW complex. There is a universal X-fibration

$$X \to UE \to \mathrm{BAut}(X).$$

inducing a one-to-one correspondence given by pull-backs of fibrations:

$$\{\textit{equivalence classes of } X \textit{ fibrations over } B\} \longleftrightarrow [B, \mathrm{BAut}(X)].$$

$\mathrm{Aut}(X)$ in Rational Homotopy Theory

Rational Fibrations

A fibration $F\colon E \to B$ of simply connected spaces has a *relative Sullivan model*: $\wedge W, \delta \to \wedge W \otimes \wedge V, D \to \wedge V, d$
with $D|W = \delta$ and $D(v) = d(v) + \sum_j w_j\theta_j(v) + \Phi$ for
$\Phi \in \wedge^+ W \cdot \wedge^+ W \otimes \wedge V$ and θ_j a *derivation* of $\wedge V$ of degree $|w_j|$

Derivation DG Lie Algebra

Let $\wedge V, d$ = DG algebra model for a simply connected space X. Define

$$\mathrm{Der}_m(\wedge V) = \{\theta\colon\ \wedge^n V \to \wedge^{n-m} V \mid \theta(xy) = \theta(x)y + (-1)^{m|x|}x\theta(y)\}$$

$$\text{bracket} : [\theta, \varphi] = \theta \circ \varphi - (-1)^{|\theta||\varphi|}\varphi \circ \theta$$

$$\text{differential} : D(\theta) = [d, \theta] = d \circ \theta - (-1)^{|\theta|}\theta \circ d$$

Theorem (Sullivan, 1977)

Let X be a simply connected CW complex. Then

$$H_*(\mathrm{Der}(\wedge V), D) \cong \pi_*(\mathrm{Aut}(X_{\mathbb{Q}})).$$

Examples

- $\mathrm{Aut}(S^{2n+1}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2n+1)$
- $\mathrm{Aut}(S^{2n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n-1)$
- $\mathrm{Aut}(\mathbb{C}P^n) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 3) \times \cdots \times K(\mathbb{Q}, 2n+1)$

Realization Problem

Given a space Y find a space X so that $\mathrm{Aut}(X_{\mathbb{Q}}) \simeq \Omega Y_{\mathbb{Q}}$.

Example (Lupton-S., 2016)

For each $n \geq 2$ and there exists a π-finite space X^n with

$$\mathrm{Aut}(X^n_{\mathbb{Q}}) \simeq_{\mathbb{Q}} \Omega(S^{2n+1} \times S^{2n+3}).$$

Homotopy-Finite Spaces

Definition

Say X is π-finite if $\pi_n(X) \otimes \mathbb{Q} = 0$ for $n \geq N$ for some N. If N is maximal, say N is the top homotopy degree for X.

Theorem

If X is π-finite with top homotopy degree N then $\mathrm{Aut}(X)$ is π-finite with top homotopy degree N, as well. Further, $\pi_N(\mathrm{Aut}(X)) \otimes \mathbb{Q} \cong \pi_N(X) \otimes \mathbb{Q}$.

Refined Inverse Problem

Given a π-finite space Y is there a π-finite space X with $\mathrm{Aut}(X) \simeq_{\mathbb{Q}} \Omega Y$.?

A Negative Result

Theorem

The space $\Omega\mathbb{C}P^2$ cannot be realized, up to rational homotopy type, as $\mathrm{Aut}(X)$ for X simply connected and π-finite.

Sketch of Proof:

Suppose X is π-finite, 1-connected with $\mathrm{Aut}(X) \simeq_{\mathbb{Q}} \Omega\mathbb{C}P^2$. Recall

$$\pi_k(\Omega\mathbb{C}P^2) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{for} \quad k = 1, 4 \\ 0 & \text{otherwise} \end{cases}$$

The minimal model for X must be of the form:

$$\wedge(v_1, \ldots, v_r, u_1, \ldots, u_s, y), d \quad \text{with} \quad |v_i| = 2, |u_j| = 3, |y| = 4$$

Step 1: The derivations (y, v_i) and $(v_i, 1)$ are cycles and so must be boundaries. Conclude that, in the model for X:

$$\wedge(v_1, \ldots, v_r, u_1, \ldots, u_s, y), d \quad \text{with} \quad |v_i| = 2, |u_j| = 3, |y| = 4$$

we have $r = s \geq 2$ and $d(y) = u_1 v_1 + \cdots + u_r v_r$.

Step 2: Let z have $|z| = 2$. Construct an inclusion of models:

$$\wedge(z), 0 \to \wedge(z) \otimes \wedge(v_1, \ldots, v_r, u_1, \ldots, u_r, y), D$$

$$\text{with} \quad D(u_1) = zv_2 + d(u_1) \qquad D(u_2) = -zv_1 + d(u_2)$$

and $D = d$ for all other generators.

Check that

$$D^2(y) = D(u_1 v_1 + u_2 v_2 + \cdots u_r v_r)$$
$$= zv_2 v_1 - zv_1 v_2 + d^2(y) = 0.$$

This is a rational fibration.

Step 3. The relative Sullivan model:

$$\wedge(z), 0 \to \wedge(z) \otimes \wedge V, D \to \wedge V, d$$

has spatial realization:

$$X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to K(\mathbb{Q}, 2).$$

This $X_{\mathbb{Q}}$-fibration is non-trivial. It must then has a rationally essential classifying map:

$$h\colon K(\mathbb{Q}, 2) \to \mathrm{BAut}(X_{\mathbb{Q}})$$

Since $H^*(K(\mathbb{Q}, 2) \cong \wedge(z)$ is a polynomial algebra it follows that $H^*(\mathrm{BAut}(X_{\mathbb{Q}}); \mathbb{Q})$ has infinite cup-length. But $\mathbb{C}P^2$ has cup-length $= 2$. Therefore

$$\mathrm{BAut}(X_{\mathbb{Q}}) \not\simeq_{\mathbb{Q}} \mathbb{C}P^2.$$

Thank you!!